United Kingdom
Mathematics Trust

Senior Mathematical Challenge 2019

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Solutions

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	B	A	B	B	D	D	B	D	C	C	E	D	C	B	A	E	E	A	A	D	C	B	E	A

1. What is the value of $123^2 - 23^2$?

A 10 000 B 10 409 C 12 323 D 14 600 E 15 658

Solution **D**

The value of $123^2 - 23^2 = (123 - 23)(123 + 23) = 100 \times 146 = 14600$.

Hints

Rather than direct calculation here, consider the more elegant method of using the difference of two squares $x^2 - y^2 = (x - y)(x + y)$.

2. What is the value of $(2019 - (2000 - (10 - 9))) - (2000 - (10 - (9 - 2019)))$?

A 4040 B 40 C −400 D −4002 E −4020

Solution **B**

The value of $(2019-(2000-(10-9)))-(2000-(10-(9-2019))) = (2019-1999)-(2000-2020) = 20 - (-20)$ which equals 40.

Hints

Work carefully from the innermost brackets to the outermost.

3. Used in measuring the width of a wire, one mil is equal to one thousandth of an inch. An inch is about 2.5 cm.

Which of these is approximately equal to one mil?

A $\frac{1}{40}$ mm B $\frac{1}{25}$ mm C $\frac{1}{4}$ mm D 25 mm E 40 mm

Solution **A**

One mil $= \frac{1}{1000}$ in $\approx \frac{1}{1000} \times 2.5$ cm $= \frac{25}{1000}$ mm $= \frac{1}{40}$ mm.

Hints

Start by writing the words of the statement in symbolic form.

4. For how many positive integer values of n is $n^2 + 2n$ prime?

A 0 B 1 C 2 D 3 E more than 3

Solution **B**

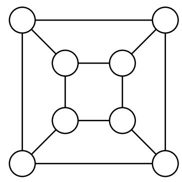
The expression $n^2 + 2n$ factorises to $n(n+2)$. For $n(n+2)$ to be prime, one factor must equal 1 whilst the other must be equal to a prime. This happens when $n = 1$, as $n + 2 = 3$, but not when $n + 2 = 1$ as n would be negative. There is therefore exactly one positive integer value of n which makes $n^2 + 2n$ prime.

Hints

Consider the definition of a prime, in terms of its factors. Try and write the given expression in the same form.

5. Olive Green wishes to colour all the circles in the diagram so that, for each circle, there is exactly one circle of the same colour joined to it.

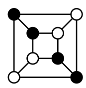
What is the smallest number of colours that Olive needs to complete this task?



A 1 B 2 C 3 D 4 E 5

Solution **B**

Each circle in the diagram is connected to three others, exactly one of which must be filled with the same colour. So, the number of colours required is greater than 1. One possible colouring with just two colours is shown here.



Hints

Consider the symmetry of the diagram.

6. Each of the factors of 100 is to be placed in a 3 by 3 grid, one per cell, in such a way that the products of the three numbers in each row, column and diagonal are all equal. The positions of the numbers 1, 2, 50 and x are shown in the diagram.

x	1	50
2		

What is the value of x?

A 4 B 5 C 10 D 20 E 25

Solution **D**

The product of all the factors of 100 is $1 \times 100 \times 2 \times 50 \times 4 \times 25 \times 5 \times 20 \times 10 = 1\,000\,000\,000$. As there are three rows, each of which has the same 'row product', that row product is 1000. So, considering the top row, $x \times 1 \times 50 = 1000$ and therefore $x = 20$. The completed grid is as shown.

20	1	50
25	10	4
2	100	5

Hints

Start by making a complete list of the factors of 100. Consider the required 'row product'.

7. Lucy is asked to choose p, q, r and s to be the numbers 1, 2, 3 and 4, in some order, so as to make the value of $\frac{p}{q} + \frac{r}{s}$ as small as possible.

What is the smallest value Lucy can achieve in this way?

A $\frac{7}{12}$ B $\frac{2}{3}$ C $\frac{3}{4}$ D $\frac{5}{6}$ E $\frac{11}{12}$

Solution **D**

In order to minimise the value of $\frac{p}{q} + \frac{r}{s}$ we need to make p and r as small as possible and make q and s be as large as possible. Considering $\frac{1}{3} + \frac{2}{4}$ and $\frac{1}{4} + \frac{2}{3}$ and then removing both $\frac{1}{3}$ and $\frac{1}{4}$ from each sum leaves the first with value $\frac{1}{4}$ and the second with value $\frac{1}{3}$. As $\frac{1}{4} < \frac{1}{3}$, the first sum has the smallest value, which is $\frac{5}{6}$.

Hints

Consider the sensible positions in which to put the smaller numbers and the larger numbers.

8. The number x is the solution to the equation $3^{(3^x)} = 333$.

Which of the following is true?

A $0 < x < 1$ B $1 < x < 2$ C $2 < x < 3$ D $3 < x < 4$ E $4 < x < 5$

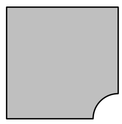
Solution **B**

Considering some integer powers of 3, we have $3^5 = 243$ and $3^6 = 729$. As $243 < 333 < 729$, $3^5 < 3^{(3^x)} < 3^6$ implies that $5 < 3^x < 6$. Rewriting once again to include powers of 3, gives $3^1 < 5 < 3^x < 6 < 3^2$, so $3^1 < 3^x < 3^2$ and finally, $1 < x < 2$.

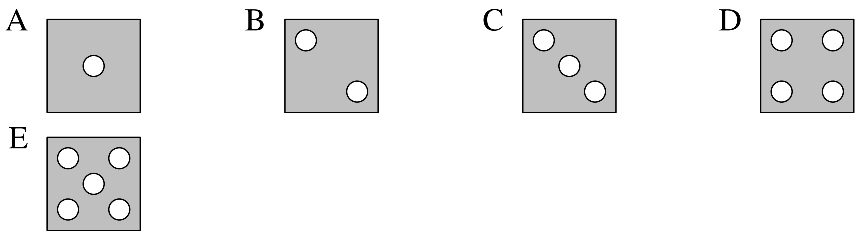
Hints

Start by thinking about the size of some small powers of three.

9. A square of paper is folded in half four times to obtain a smaller square. Then a corner is removed as shown.



Which of the following could be the paper after it is unfolded?



Solution **D**

Folding the paper four times gives 2^4 layers. Removing a corner, 16 quarter-circles are formed. Of the given options, only D, with four whole circles could then be possible.

Hints

We cannot tell from the question or the diagram where the fold lines are, so is there a way to tackle this without that being important?

10. Which of the following five values of n is a counterexample to the statement in the box below?

For a positive integer n, at least one of $6n - 1$ and $6n + 1$ is prime.

A 10 B 19 C 20 D 21 E 30

Solution **C**

To provide a counterexample, we are looking for both the values of $6n - 1$ and $6n + 1$ to be composite for a particular n . When $n = 20$, $6n - 1 = 119 = 7 \times 17$ and $6n + 1 = 121 = 11 \times 11$, which is our counter-example. In each of the other cases, at least one value is prime.

Hints

Consider what properties a positive integer has if it is not prime.

11. For how many integer values of k is $\sqrt{200 - \sqrt{k}}$ also an integer?

A 11 B 13 C 15 D 17 E 20

Solution **C**

In order for $\sqrt{200 - \sqrt{k}}$ to be an integer, $200 - \sqrt{k}$ must be a square. As $\sqrt{k} \geq 0$, the smallest possible value of $200 - \sqrt{k}$ which is square is 0, (when $k = 200^2$) and the largest is $14^2 = 196$, (when $k = 4^2$). Counting the squares from 0^2 to 14^2 gives 15 values.

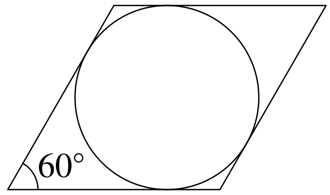
Hints

Consider the smallest and largest values of k.

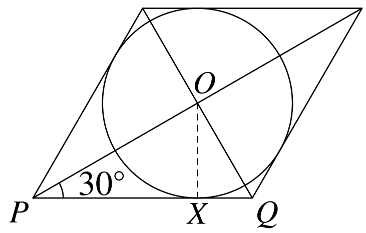
12. A circle with radius 1 touches the sides of a rhombus, as shown. Each of the smaller angles between the sides of the rhombus is 60°.

What is the area of the rhombus?

A 6 B 4 C $2\sqrt{3}$ D $3\sqrt{3}$ E $\frac{8\sqrt{3}}{3}$



SOLUTION **E**



Let the centre of the rhombus and circle be O. Let two of the vertices along an edge of the rhombus be P and Q and let X be the point on PQ where the rhombus is tangent to the circle. In order to relate the radius of the inscribed circle to a useful measurement on the rhombus, we can split the rhombus along its diagonals into four congruent triangles, one of which is POQ. As PQ is tangent to the circle at X, PQ and OX are perpendicular. Triangles OXP and OXQ are then similar 30°, 60°, 90° triangles with $OX = 1$, $XP = \sqrt{3}$ and $XQ = \frac{1}{\sqrt{3}}$. The area of the rhombus is then $4 \times \frac{1}{2} \times (\sqrt{3} + \frac{1}{\sqrt{3}}) \times 1 = 2 \times \frac{(3+1)}{\sqrt{3}} = \frac{8}{3}\sqrt{3}$.

HINTS

Try and split the rhombus into useful right-angled triangles.

13. Anish has a number of small congruent square tiles to use in a mosaic. When he forms the tiles into a square of side n, he has 64 tiles left over. When he tries to form the tiles into a square of side $n + 1$, he has 25 too few.

How many tiles does Anish have?

A 89 B 1935 C 1980 D 2000 E 2019

Solution **D**

Anish has both $n^2 + 64$ and $(n + 1)^2 - 25$ tiles. So, $n^2 + 64 = (n + 1)^2 - 25$ which simplifies to $n^2 + 64 = n^2 + 2n + 1 - 25$ and then $64 = 2n - 24$. So $n = \frac{88}{2} = 44$. Therefore Anish has $44^2 + 64 = 1936 + 64 = 2000$ tiles.

Hints

Start by writing the words of each statement in symbolic form.

14. One of the following is the largest square that is a factor of 10!. Which one?

Note that, $n! = 1 \times 2 \times 3 \times \cdots \times (n - 1) \times n$.

A $(4!)^2$ B $(5!)^2$ C $(6!)^2$ D $(7!)^2$ E $(8!)^2$

Solution **C**

For a square to be a factor of 10!, the prime factors of the square must be present in 10! an even number of times. Writing 10! first as $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ and then as $2 \times 5 \times 3 \times 3 \times 2 \times 2 \times 2 \times 7 \times 2 \times 3 \times 5 \times 2 \times 2 \times 3 \times 2 = 2^8 \times 3^4 \times 5^2 \times 7 = (2^4 \times 3^2 \times 5)^2 \times 7$ we can see that the largest square is $2^4 \times 3^2 \times 5$. Expanding $2^4 \times 3^2 \times 5$ as $(2 \times 3) \times 5 \times (2 \times 2) \times 3 \times 2 \times 1$ shows that it is exactly 6!. So $(6!)^2$ is the largest square which is a factor of 10!.

Hints

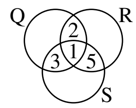
Consider how 10! is composed. What does it mean in terms of primes for a square to be a factor of a number?

15. The highest common factors of all the pairs chosen from the positive integers Q, R and S are three different primes.

What is the smallest possible value of $Q + R + S$?

A 41 B 31 C 30 D 21 E 10

Solution **B**



For $Q + R + S$ to be as small as possible, we want the highest common factors of the pairs to be as small as possible, and prime. Therefore the highest common factors are 2, 3 and 5 in some order and then Q, R and S are 2×3, 2×5 and 3×5, i.e. 6, 10 and 15, in some order. This gives $Q + R + S = 6 + 15 + 10 = 31$.

Hints

Start by considering which primes to have as the highest common factors.

16. The numbers x, y and z satisfy the equations $9x + 3y - 5z = -4$ and $5x + 2y - 2z = 13$.

What is the mean of x, y and z?

A 10 B 11 C 12 D 13 E 14

Solution **A**

As $9x + 3y - 5z = -4$ and $5x + 2y - 2z = 13$, subtracting the first equation from twice the second gives $(10x + 4y - 4z) - (9x + 3y - 5z) = 2 \times 13 - (-4)$. The mean of x, y and z which is $\frac{(x+y+z)}{3}$ is therefore $\frac{30}{3} = 10$.

Hints

Consider the shape of the algebraic expression which will allow you to find the mean of x, y and z. Do you need to know the values of each of x, y and z?

17. Jeroen writes a list of 2019 consecutive integers. The sum of his integers is 2019.

What is the product of all the integers in Jeroen's list?

A 2019^2 B $\frac{2019 \times 2020}{2}$ C 2^{2019} D 2019 E 0

Solution **E**

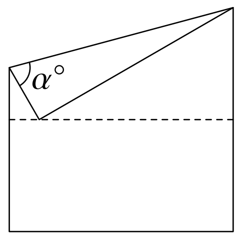
The sum of the first 2019 positive integers is $\frac{2019 \times 2020}{2}$ which is considerably larger than the required sum of 2019. In order for the sum of Jeroen's 2019 integers to be only 2019, some of the integers must be positive and some must be negative. One of the integers will then be 0, so the product will also be 0. Jeroen's list is $-1008, \ldots -2, -1, 0, 1, 2, \ldots, 1008, 1009, 1010$.

Hints

Think about what an integer is.

18. Alison folds a square piece of paper in half along the dashed line shown in the diagram. After opening the paper out again, she then folds one of the corners onto the dashed line.

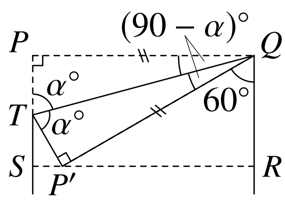
What is the value of α?



A 45 B 60 C 65 D 70 E 75

Solution **E**

Let P and Q be the vertices on the top edge of the square. Let R and S be the points at the end of the first fold line. Let P' be the position of P on the first fold line after the second fold has been made. Let T be the point on PS which lies on the second fold line. Triangles PQT and $P'QT$ are then congruent so $\angle PQT = \angle P'QT = 90° - \alpha°$. As $PQ = 2QR$ then $P'Q = 2QR$ and so $\angle P'QR = 60°$. Considering angles at Q gives $2(90 - \alpha) + 60 = 90$, so $150 = 2\alpha$ and $\alpha = 75$.

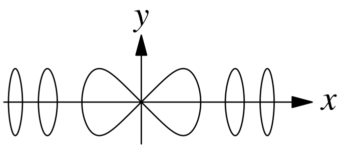


Hints

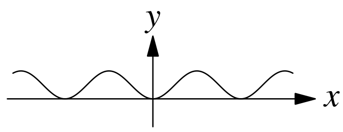
Consider which angles you know in the top half of the diagram. Don't lose sight of the fact that you are dealing with a square.

19. Which of the following could be the graph of $y^2 = \sin\left(x^2\right)$?

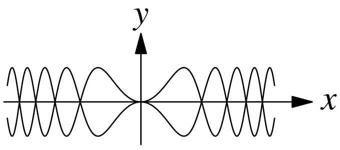
A



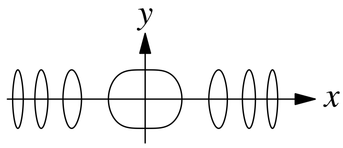
B



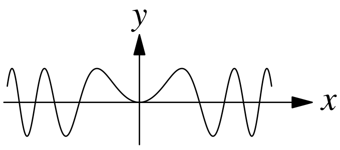
C



D



E



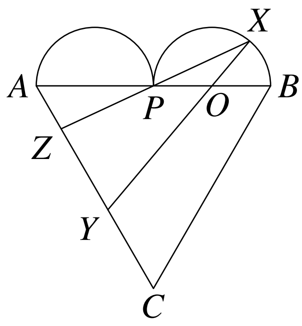
Solution **A**

As $\sin(0^2) = 0^2$ our graph must pass through the origin, so eliminating option D. As $y^2 = \sin(x^2)$, $y = \pm\sqrt{\sin(x^2)}$ and so the x-axis must be a line of symmetry, so eliminating options B and E. For some x-values, $\sin(x^2)$ will be negative and so there will be no corresponding y-value, so eliminating option C, which has y-values for every x-value. The only possible remaining option is then A and it can be checked that the graph is indeed of this form.

Hints

Work to eliminate impossible options.

20. The "heart" shown in the diagram is formed from an equilateral triangle ABC and two congruent semicircles on AB. The two semicircles meet at the point P. The point O is the centre of one of the semicircles. On the semicircle with centre O, lies a point X. The lines XO and XP are extended to meet AC at Y and Z respectively. The lines XY and XZ are of equal length.

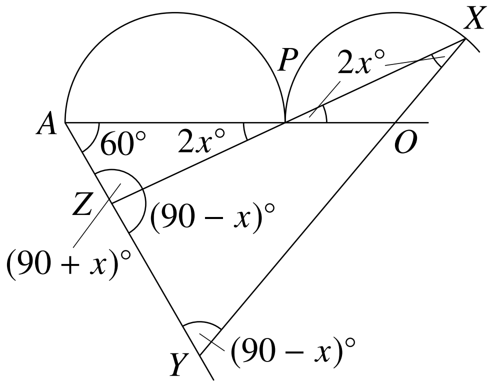


What is $\angle ZXY$?

A 20° B 25° C 30° D 40° E 45°

Solution **A**

Let $\angle ZXY = 2x°$, then the equal angles in isosceles triangle ZXY, are each $\frac{(180-2x)°}{2} = (90 - x)°$. We can then find each of the angles inside triangle AZP in terms of x. Considering angles at Z gives $\angle AZP = 180° - (90 - x)° = (90 + x)°$. Then, as triangle OXP is isosceles, $\angle OPX = \angle OXP = 2x°$. As $\angle ZPA$ is vertically opposite $\angle OPX$, $\angle ZPA$ is also equal to $2x°$. Finally, $\angle OAY = 60°$ as triangle BAC is given to be equilateral. In triangle AZP, $60 + 90 + x + 2x = 180$, so $x = 10$ and $\angle ZXY = 2x° = 20°$.



Hints

Label the angle ZXY as $2x°$ to avoid fractional angles as you 'angle chase' around the diagram.

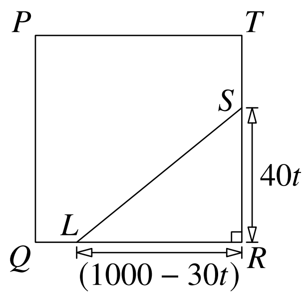
21. In a square garden $PQRT$ of side 10 m, a ladybird sets off from Q and moves along edge QR at 30 cm per minute. At the same time, a spider sets off from R and moves along edge RT at 40 cm per minute.

What will be the shortest distance between them, in metres?

A 5 B 6 C $5\sqrt{2}$ D 8 E 10

Solution **D**

Let L be the position of the ladybird on QR and let S be the position of the spider on RT each after t minutes. The shortest distance between L and S is along the straight line which is the hypotenuse of the right-angled triangle LRS. The distance QL is $30t$ cm, so the distance LR is $(1000 - 30t)$ cm. Also, the distance RS is $40t$ cm. So, $LS^2 = (1000 - 30t)^2 + (40t)^2$ which expands and simplifies to $2500(t^2 - 24t + 400) = 2500((t-12)^2 + 256)$. The distance from L to S is shortest when $t = 12$ and is $\sqrt{2500 \times 256}$ cm $= 800$ cm $= 8$ m.



Hints

Start by drawing a diagram which represents the positions of L and S part way through their journeys. Take care to be consistent with units when labelling the diagram.

22. A function f satisfies the equation $(n-2019)f(n) - f(2019-n) = 2019$ for every integer n.

What is the value of $f(2019)$?

A 0 B 1 C 2018×2019 D 2019^2 E 2019×2020

Solution **C**

At the start, taking $n = 0$ gives $(0-2019)f(0) - f(2019-0) = 2019$. Then $f(2019) = -2019(1 + f(0))$. To find $f(0)$, let $n = 2019$, then $(2019 - 2019)f(2019) - f(2019 - 2019) = 2019$, so $0 - f(0) = 2019$ and $f(0) = -2019$. Then we have $f(2019) = -2019(1 + (-2019)) = 2018 \times 2019$.

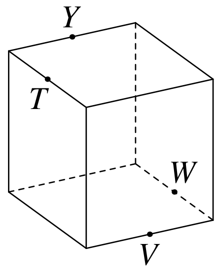
Hints

Think about which values of n will give you useful simplified expressions.

23. The edge-length of the solid cube shown is 2. A single plane cut goes through the points Y, T, V and W which are midpoints of the edges of the cube, as shown.

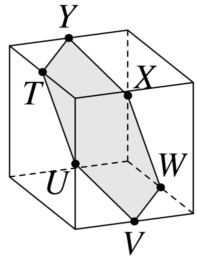
What is the area of the cross-section?

A $\sqrt{3}$ B $3\sqrt{3}$ C 6 D $6\sqrt{2}$ E 8



SOLUTION **B**

When the single plane cut is made through the cube, it passes through points Y, T, V and W and also points U and X which are midpoints of two of the remaining edges of the cube as shown. The cross-section is then a regular hexagon. As the side-length of the cube is 2, the distance between the midpoints of adjacent edges is $\sqrt{2}$. This is the length of each edge of the hexagon.The hexagon can be split into six equilateral triangles and so the area of the hexagon is $6 \times \frac{1}{2} \times \sqrt{2} \times \sqrt{2} \times \sin 60° = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$.



HINTS

Try and imagine making the plane cut. What shape is your cross section? Is that shape regular or irregular?

24. The numbers x, y and z are given by $x = \sqrt{12 - 3\sqrt{7}} - \sqrt{12 + 3\sqrt{7}}$, $y = \sqrt{7 - 4\sqrt{3}} - \sqrt{7 + 4\sqrt{3}}$ and $z = \sqrt{2 + \sqrt{3}} - \sqrt{2 - \sqrt{3}}$.

What is the value of xyz ?

A 1 B –6 C –8 D 18 E 12

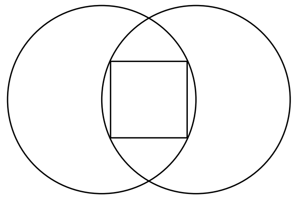
SOLUTION **E**

Each of x, y and z has the same form which is $\sqrt{a - b} - \sqrt{a + b}$ for some a and b. Squaring this expression gives $(a - b) - 2\sqrt{a + b} \times \sqrt{a - b} + (a + b) = 2a - 2\sqrt{(a^2 - b^2)}$. Applying this to x with $a = 12$ and $b = 3\sqrt{7}$ gives $x^2 = 24 - 2\sqrt{81} = 6$. Similarly we can calculate that $y^2 = 12$ and $z^2 = 2$. This gives us that $x^2y^2z^2 = 6 \times 12 \times 2 = 144$. From the initial expressions $x < 0$, $y < 0$ and $z > 0$, so $xyz > 0$ and therefore $xyz = 12$.

HINTS

Look for similarities in the form of each of x, y and z. What expression would be easier to calculate than xyz ?

25. Two circles of radius 1 are such that the centre of each circle lies on the other circle. A square is inscribed in the space between the circles.

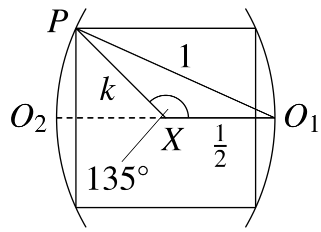


What is the area of the square?

A $2 - \frac{\sqrt{7}}{2}$ B $2 + \frac{\sqrt{7}}{2}$ C $4 - \sqrt{5}$

D 1 E $\frac{\sqrt{5}}{5}$

Solution **A**



Let the centres of the circles and the square be O_1, O_2 and X respectively. Let P be the point on the circle with centre O_1 which is a vertex of the square. Then $O_1P = 1$ and $O_1O_2 = 1$ so $O_1X = \frac{1}{2}$. Let $XP = k$, so the area of the square will be $2k^2$. As O_1X is parallel to the top edge of the square and XP goes from the centre of the square to a vertex, angle $O_1XP = 135°$. Using the cosine rule on triangle O_1XP gives $1^2 = (\frac{1}{2})^2 + k^2 - 2 \times \frac{1}{2} \times k \times \cos 135°$. As $\cos 135° = -\cos 45° = -\frac{1}{\sqrt{2}}$, this simplifies to $4k^2 + 2\sqrt{2}k - 3 = 0$. Since we know $k > 0$, $k = \frac{-1+\sqrt{7}}{2\sqrt{2}}$. So $\sqrt{2}k = \frac{-1+\sqrt{7}}{2}$ and the area of the square is $2k^2 = 2 - \frac{\sqrt{7}}{2}$.

Hints

Consider a triangle whose vertices are the centre of the square, a vertex of the square and the centre of the circle furthest from that vertex.